



# Double General Point Interactions: Symmetry and Tunneling Times

Molly A. Lee<sup>1</sup>, José T. Lunardi<sup>2,3</sup>, Luiz A. Manzoni<sup>1\*</sup> and Erik A. Nyquist<sup>1</sup>

<sup>1</sup> Department of Physics, Concordia College, Moorhead, MN, USA, <sup>2</sup> School of Physics and Astronomy, University of Glasgow, Glasgow, UK, <sup>3</sup> Departamento de Matemática e Estatística, Universidade Estadual de Ponta Grossa, Ponta Grossa, Brazil

We consider the one dimensional problem of a non-relativistic quantum particle scattering off a double barrier built from two generalized point interactions (each one characterized as a member of the four parameter family of point interactions). The properties of the double point barrier under parity transformations are investigated, using the distributional approach, and the constraints on the parameters necessary for the interaction to have a well-defined parity are obtained. We show that the limit of zero interbarrier distance of a renormalized odd arrangement with two  $\delta'$  is either trivial or does not exist as a generalized point interaction. Finally, we specialize to double barriers with defined parity, calculate the phase and Salecker-Wigner-Peres clock times and argue that the emergence of the generalized Hartman effect is an artifact of the *extreme* opaque limit.

**Keywords:** generalized point interactions, delta interaction, delta prime interaction, tunneling times, generalized Hartman effect, Salecker-Wigner-Peres clock

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### \*Correspondence:

Luiz A. Manzoni  
manzoni@cord.edu

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## 1. INTRODUCTION

The search for a consistent definition of the time it takes for a particle to traverse a classically forbidden region is an enduring problem in quantum mechanics (see, e.g., [1, 2] and references therein). The difficulties involved in the problem are both experimental and theoretical. The time scales involved, typically of the order of attoseconds, only recently became accessible experimentally [3–5], and the interpretation of the experimental results is not yet free of controversy [5, 6]. From the theoretical point of view, the difficulties are related to the impossibility of defining a self-adjoint (observable) time operator in quantum mechanics. Thus, there arises the need for operational definitions of quantum tunneling time.

Several competing definitions of tunneling time exist, among which the most well-known and useful are the phase time [7], the dwell time [8], the Larmor time [9, 10], and the Salecker-Wigner-Peres (SWP) clock time [11–14]. However, all of these tunneling times can lead to counter-intuitive results in specific situations. In particular, all of them can result in superluminal speeds for particles tunneling through an opaque barrier, the so-called Hartman effect (HE) [15], and thus their status as traversal times is controversial.

The HE can be generalized for multiple potential barriers, in which case it has been established that in the opaque limit (low probability of transmission) the phase time becomes independent of the barrier spacing [16]; this so-called generalized Hartman effect (GHE) has been verified both in non-relativistic [16–19] and relativistic [20] quantum mechanics and it is usually assumed to hold for other time scales, such as the dwell time [21]. At first sight the GHE seems to be even more troublesome than the original version of the HE, since one would expect that for the free region in between the barriers the traversal time would be well described by the phase time.

The vast literature discussing tunneling times and the HE, as well as its generalized version, focuses mostly in one-dimensional square potential barriers, with few other potentials such as the sech-square potential [22] and Dirac delta barriers (see, e.g., [21, 23, 24] and references therein) having been considered. From a practical point of view, the understanding of one-dimensional tunneling may help to understand phenomena in nanoscale systems such as, for example, conductance in nanowires (see [25] and references therein). However, to reach a better understanding of tunneling times and the associated questions, it is important and useful to study a larger variety of potentials, such as the generalized one-dimensional point interactions.

Point interactions in one-dimensional quantum mechanics have been the source of significant theoretical and mathematical interest for providing the opportunity to investigate regularization and renormalization schemes in the framework of quantum mechanics (see, e.g., [26–29]). There have also been controversies in the literature due to conflicting regularization-dependent results obtained in the treatment of the so-called  $\delta'$ -interaction [30–33]. These controversies, however, were clarified by using mathematically rigorous methods such as self-adjoint extensions (SAE) [34–37] and Schwartz's distribution theory [38, 39] (there are also alternative approaches using non-standard distribution theory, e.g., [40, 41]), both of which unequivocally demonstrate the existence of a four-parameter family of point interactions in one dimensional quantum mechanics—these include, but are not restricted to, the  $\delta$  interaction as well as other interactions commonly associated with the  $\delta'$  in the literature. It should be noticed that the distributional approach [39] makes it unambiguously clear, from parity considerations, that a “genuine”  $\delta'$  interaction does not exist (for a *particular regularization* of the  $\delta'$  interaction displaying *even* behavior under parity transformations see [42]).

From the point of view of applications, point-like interactions have long provided models of significant experimental relevance for lower dimensional systems [43, 44], such as the Girardeau-Tonks gas [45, 46] and the Lieb-Liniger model [47, 48] and, more recently, as a zero-range model of ultra-thin heterostructures [49, 50]. The theoretical study of tunneling across point barriers is another important application of these interactions, since it has the important feature that one does not need to be concerned with above-the-barrier components when considering tunneling of wave packets, while for barriers of finite height such components dominate the transmitted packet in the case of opaque barriers and lead to difficulties in the interpretation of tunneling times—tunneling across point barriers is, therefore, free of such difficulties.

The general four-parameter family of interactions in one dimension has been used to investigate tunneling times for a non-relativistic particle incident on a single point scatterer [51]. However, in order to generate a richer tunneling structure and to investigate some of the more counterintuitive phenomena associated with tunneling times it is necessary to consider (at least) two point scatterers. For example, it is not difficult to convince oneself that for a single point barrier the SWP clock time (and the dwell time) must be trivially zero, while for two point barriers it is non-trivial. Furthermore, in addition to its

relevance to the study of resonant tunneling in heterostructures in the zero-range limit [49, 50], the presence of two point barriers allows us to investigate the GHE—in contradistinction, the HE cannot be addressed by considering only one point scatterer, since the distance “traveled” in that case vanishes (the support of a group of point interactions is a set of zero measure). Therefore, in this work we investigate the scattering of a non-relativistic particle by two point scatterers given by the most general potential [34–36, 39]. Given the relevance of parity-symmetric considerations in the controversial discussion of regularized implementations of point interactions, and the related discussion of the  $\delta'$ -interaction, we start by introducing the singular interaction following the distributional approach introduced in Lunardi and Manzoni [39], and analyzing the behavior of such potential under parity transformations, thus obtaining the necessary conditions to have even and odd potentials under such transformations. We also analyze the limits in which the separation between the barriers shrinks to zero and extend the work of Šeba [35] to include the limit for a *renormalized*<sup>1</sup> odd-arrangement of two  $\delta'$ -interactions<sup>2</sup>. Then, we proceed to calculate the phase and SWP clock times for barriers with well defined parity symmetry, analyze the behavior of these times in the opaque regime and discuss whether GHE occurs or not for these potentials (a preliminary report on tunneling times, not accounting for the symmetry of the interactions, appeared in Lee et al. [52]).

## 2. DOUBLE BARRIER OF GENERALIZED POINT INTERACTIONS: SYMMETRY UNDER PARITY

Let us consider the scattering of a non-relativistic particle of mass  $m$  and energy  $E = \frac{\hbar^2 k^2}{2m}$  by two general point interactions localized at  $x = x_1$  and  $x = x_2$ , with  $x_1 < x_2$ . According to Calçada et al. [39] the distributional Schrödinger equation can be written as (throughout this paper we will adopt the atomic Rydberg units such that  $\hbar = 1$  and  $2m = 1$ , as usual when dealing with point interactions)

$$\psi''(x) + k^2 \psi = s_1 [\psi](x) + s_2 [\psi](x) \equiv s[\psi](x), \quad (1)$$

where the prime indicates a space derivative, and the interaction distributions are given by

$$s_j [\psi](x) = \left[ c_j e^{i\theta_j} \psi(x_j^-) + (d_j e^{i\theta_j} - 1) \psi'(x_j^-) \right] \delta(x - x_j) + \left[ (a_j e^{i\theta_j} - 1) \psi(x_j^-) + b_j e^{i\theta_j} \psi'(x_j^-) \right] \delta'(x - x_j) \quad (2)$$

or

$$s_j [\psi](x) = \left[ c_j e^{-i\theta_j} \psi(x_j^+) - (a_j e^{-i\theta_j} - 1) \psi'(x_j^+) \right] \delta(x - x_j) - \left[ (d_j e^{-i\theta_j} - 1) \psi(x_j^+) - b_j e^{-i\theta_j} \psi'(x_j^+) \right] \delta'(x - x_j), \quad (3)$$

<sup>1</sup>Following Šeba's nomenclature [35].

<sup>2</sup>Throughout this work the notation  $\delta'$  refers to the most widely used version of such interaction, which will be presented in Section 2 together with an alternative version denoted by  $\delta^{(1)}$ .

with  $a_j, b_j, c_j, d_j \in \mathbb{R}$ ,  $a_j d_j - b_j c_j = 1$  and  $\theta_j \in [0, \pi)$ ,  $j = 1, 2$ . In these expressions we used the notation  $\psi(x_j^\pm) = \lim_{\varepsilon \rightarrow 0^+} \psi(x_j \pm \varepsilon)$ , and similarly for  $\psi'(x_j^\pm)$ . Such interaction terms correspond to *non-separated* solutions, and will be the only type of interactions considered in this work. The above distributional Schrödinger equation implies the same boundary conditions at the point singularities as those obtained from the SAE method [39, 53] (also see [54] for a rigorous study of the boundary conditions for point interactions):

$$\begin{pmatrix} \psi(x_j^+) \\ \psi'(x_j^+) \end{pmatrix} = \Lambda_j \begin{pmatrix} \psi(x_j^-) \\ \psi'(x_j^-) \end{pmatrix}, \quad \text{with } \Lambda_j = e^{i\theta_j} \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix} \quad j = 1, 2. \quad (4)$$

The general solution for the Schrödinger equation in the region outside the double potential is

$$\psi(x) = \begin{cases} A_1 e^{ikx} + A_2 e^{-ikx}, & x < x_1 \\ C_1 e^{ikx} + C_2 e^{-ikx}, & x > x_2 \end{cases}, \quad (5)$$

where

$$\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \mathcal{M}(\Lambda_1, \Lambda_2, k, q, x_1, x_2) \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}, \quad (6)$$

with  $k$  and  $q$  being the wave numbers in the region outside and inside the double barrier, respectively, and<sup>3</sup>

$$\mathcal{M}(\Lambda_1, \Lambda_2, k, q, x_1, x_2) = M^{-1}(k, x_2) \Gamma(\Lambda_1, \Lambda_2, q, x_1, x_2) M(k, x_1), \quad (7)$$

with the transfer matrix given by:

$$\Gamma(\Lambda_1, \Lambda_2, q, x_1, x_2) = \Lambda_2 M(q, x_2) M^{-1}(q, x_1) \Lambda_1 \quad (8)$$

and

$$M(\rho, x) = \begin{pmatrix} e^{i\rho x} & e^{-i\rho x} \\ i\rho e^{i\rho x} & -i\rho e^{-i\rho x} \end{pmatrix}. \quad (9)$$

From Equations (5) and (6) we can write

$$\begin{pmatrix} \psi(x_2^+) \\ \psi'(x_2^+) \end{pmatrix} = \Gamma(\Lambda_1, \Lambda_2, q, x_1, x_2) \begin{pmatrix} \psi(x_1^-) \\ \psi'(x_1^-) \end{pmatrix}. \quad (10)$$

From the above expressions we observe that when  $x_2 \rightarrow x_1$  the double point interaction converges to a single point interaction, with the transfer matrix given by  $\Gamma = \Lambda_2 \Lambda_1$ , i.e., in this limit the composition law for the parameters has the  $U(1) \times SL(2, \mathbb{R})$  group structure (see [55] for the group structure of a restricted class of double point interactions). The explicit form of the interaction term  $s[\psi]$  in Equation (1) is suitable to study its symmetry properties [39]. To this end, and without losing generality, let us assume  $x_2 = -x_1 \equiv L > 0$ . Then, under a parity transformation this term changes as

$$s[\psi](x) \rightarrow \tilde{s}[\psi](x) = \tilde{s}_1[\psi](x) + \tilde{s}_2[\psi](x), \quad (11)$$

where

<sup>3</sup>Even though in this Section  $q = k$ , such a notation will be useful in the next Section, where we will consider a small constant potential  $V_0 \neq 0$  in the region between the barriers.

$$\begin{aligned} \tilde{s}_j[\psi](x) &= [c_j e^{i\theta_j} \psi((-x_j)^+) - (d_j e^{i\theta_j} - 1) \psi'((-x_j)^+)] \\ &\quad \delta(x + x_j) + [- (a_j e^{i\theta_j} - 1) \psi((-x_j)^+) + b_j e^{i\theta_j} \psi'((-x_j)^+)] \delta'(x + x_j), \quad j = 1, 2. \end{aligned} \quad (12)$$

The interaction is *even* if  $\tilde{s}[\psi] = s[\psi]$  and it is *odd* if  $\tilde{s}[\psi] = -s[\psi]$  (for details see [39]). These conditions, taking Equation (3) into account, imply that the double general interactions with well defined parity symmetry are characterized by the following constraints:

– *Even Interactions*

$$\theta_2 = -\theta_1, \quad a_2 = d_1, \quad b_2 = b_1, \quad c_2 = c_1 \quad \text{and} \quad d_2 = a_1, \quad (13)$$

which is equivalent to the two b.c. matrices (redefining the parameters as  $\theta_1 \equiv \theta$ ,  $d_1 \equiv d$ ,  $b_1 \equiv b$ ,  $c_1 \equiv c$  and  $a_1 \equiv a$ )

$$\Lambda_1^{(e)} = e^{i\theta} \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \Lambda_2^{(e)} = e^{-i\theta} \begin{pmatrix} d & b \\ c & a \end{pmatrix}, \quad \text{with } ad - bc = 1. \quad (14)$$

From the above, the even arrangements of two general point interactions constitute a four parameter family of interactions.

– *Odd Interactions*

$$\theta_1 = \theta_2 = 0, \quad a_2 = 2 - d_1, \quad b_2 = -b_1, \quad c_2 = -c_1, \quad d_2 = 2 - a_1, \quad (15)$$

and the conditions  $a_i d_i - b_i c_i = 1$  ( $i = 1, 2$ ) further imply  $a_2 = a_1$  (thus  $d_2 = d_1$ ), with  $b_1 c_1 \leq 0$ .

Redefining  $b_1 \equiv b$  and  $c_1 \equiv c$ , odd interactions can be conveniently characterized by the b.c. matrices

$$\begin{aligned} \Lambda_1^{(o)} &= \begin{pmatrix} 1 + s\sqrt{-bc} & b \\ c & 1 - s\sqrt{-bc} \end{pmatrix}, \\ \Lambda_2^{(o)} &= \begin{pmatrix} 1 + s\sqrt{-bc} & -b \\ -c & 1 - s\sqrt{-bc} \end{pmatrix}, \end{aligned} \quad (16)$$

with  $bc \leq 0$  and  $s = \pm 1$ . Thus, the odd arrangements of two general point interactions constitute a three parameter family of interactions.

From Equation (14) one can see that, given an arbitrary point interaction, it is always possible to find a suitable second point interaction resulting on an even arrangement<sup>4</sup>; on the other hand, odd double point interactions are more restrictive and can only be built by combining very specific kinds of point interactions, as revealed by the constraints on the parameters of the two b.c. matrices indicated in Equation (16). Particularly interesting among the even arrangements of double-interactions are those built from two delta ( $\delta$ ) interactions ( $\theta = 0$ ,  $a =$

<sup>4</sup>The parity symmetric behavior of the even arrangements of  $\delta$  and  $\delta'$  could be inferred from the symmetries of quantum graphs considered in Boman and Kurasov [56] and also from the investigation of  $\mathcal{PT}$ -Hermiticity in Albeverio et al. [57]. However, our treatment applies to a more general class of interactions which include the parity symmetric properties of arrangements of single point interactions without well defined symmetry (e.g., the even arrangement of two  $\delta^{(1)}$ ) and, in particular, it allows for odd arrangements of two point interactions.

$d = 1, b = 0, c$  arbitrary), two “delta prime” ( $\delta'$ ) interactions ( $\theta = 0, a = d = 1, c = 0, b$  arbitrary) and two  $\delta^{(1)}$  interactions ( $\theta = 0, b = c = 0, a = d^{-1}$  arbitrary). Among the odd arrangements we are interested in those built from two  $\delta$  or two  $\delta'$  (it is not possible to build an odd arrangement from two  $\delta^{(1)}$  interactions). As already mentioned in Section 1, neither  $\delta'$  nor  $\delta^{(1)}$  stand for a “genuine” derivative of a  $\delta$  interaction—they simply refer to particular choices of parameters in the general point interaction [34–36] (a recent proposal for physically interpreting the boundary conditions given by single point interactions is given in Kulinskii and Panchenko [58]).

It is well known that single point interactions are either even under parity or have no defined symmetry, that is, there exists no single point interaction which is odd under parity [39] (for instance, both a single  $\delta$  and a single  $\delta'$  are even interactions, whereas a single  $\delta^{(1)}$  has no defined parity). Thus, there naturally arises the question: what is the resulting symmetry of an interaction obtained by shrinking the inter-barrier spacing to zero for an initial double point barrier with well defined *odd* symmetry? It is simple to show that in this limit, characterized by taking  $x_2 \rightarrow x_1$  in Equation (8), the resulting interaction either does not have a defined symmetry or vanishes (when  $bc = 0$ ), a result in accordance with the conclusion that there is no possible regularization capable of producing an odd single point interaction [39]. The equivalent limit for an even double point barrier results, unsurprisingly, in an *even* point interaction.

Šeba [35] investigated the limit  $L \rightarrow 0$  of the *renormalized* dipole interaction  $(1/2L^\nu)[\delta(x+L) - \delta(x-L)]$ , in the context of SAE, and found that for  $\nu < \frac{1}{2}$  this renormalized *odd* interaction converges to the free case, for  $\nu = \frac{1}{2}$  it converges to the (*even*) delta interaction and for  $\nu > \frac{1}{2}$  such a limit results in separated solutions (i.e., free Hamiltonians on the two half-lines defined from the origin). Šeba’s results can be immediately reproduced in the formalism developed above by assuming the parameters for an odd arrangement of delta interactions in Equation (16),  $b = 0$  and  $c \equiv \lambda L^{-\nu}$ , and taking the limit  $L \rightarrow 0$  of  $\Gamma_{\delta'}^\nu(\Lambda_1^{(o)}, \Lambda_2^{(o)}, k, -L, L)$ , where we introduced the indices  $\delta$  and  $\nu$  to indicate the odd arrangement of deltas and the dependence on the parameter  $\nu$ , respectively.

Let us now extend the above analysis to the case of a renormalized odd arrangement of “delta prime” interactions. Thus, choosing  $b \equiv \lambda L^{-\nu}$  and  $c = 0$  in Equation (16) we obtain

$$\Gamma_{\delta'}^\nu(\Lambda_1^{(o)}, \Lambda_2^{(o)}, k, -L, L) \simeq \begin{pmatrix} 1 + 2\lambda k^2 L^{1-\nu} & 2\lambda^2 k^2 L^{1-2\nu} \\ -2k^2 L & 1 - 2\lambda k^2 L^{1-\nu} \end{pmatrix}; \quad \nu > 0, \quad (17)$$

where we followed the same notation of the previous paragraph and the subscript  $\delta'$  indicates the odd arrangement of “delta prime” interactions. From Equation (17) we have the limits

$$\lim_{L \rightarrow 0} \Gamma_{\delta'}^{\nu < 1/2}(\Lambda_1^{(o)}, \Lambda_2^{(o)}, k, -L, L) = \mathbf{1}; \quad (18)$$

$$\lim_{L \rightarrow 0} \Gamma_{\delta'}^{\nu = 1/2}(\Lambda_1^{(o)}, \Lambda_2^{(o)}, k, -L, L) = \begin{pmatrix} 1 & 2\lambda^2 k^2 \\ 0 & 1 \end{pmatrix}; \quad (19)$$

$$\lim_{L \rightarrow 0} \Gamma_{\delta'}^{1/2 < \nu < 1}(\Lambda_1^{(o)}, \Lambda_2^{(o)}, k, -L, L) = \begin{pmatrix} 1 & h k^2 \\ 0 & 1 \end{pmatrix}, \quad h \equiv +\infty; \quad (20)$$

where  $\mathbf{1}$  indicates the  $2 \times 2$  identity matrix. For  $\nu \geq 1$ , similarly to Equation (20), the limit will result in matrices whose elements have improper limits [albeit different from Equation (20)].

From Equations (18) to (20), it follows that the renormalized odd arrangement of  $\delta'$  interactions converges to the free case for  $\nu < \frac{1}{2}$ . For  $\nu = \frac{1}{2}$  the limit (Equation 19) seems, at first, to result in a single (*even*)  $\delta'$  interaction. However, the  $k^2$ -dependence in the limiting transfer matrix would imply an interaction distribution with singular order  $+3$  in Equation (1) and would violate the tenets of the distributional approach, which allows a maximum singular order of  $+1$  [39]—otherwise one would have singular wave functions from which no physically meaningful solutions of the Schrödinger equation can be obtained<sup>5</sup>. Therefore, interactions such as Equation (19) are not allowed by the distributional approach to general singular interactions (nor are they allowed by the SAE method), and for  $\nu = \frac{1}{2}$  the zero range limit of a renormalized odd arrangement of  $\delta'$  interactions does not converge to a well defined (in the sense of distributions) singular interaction. In the same way, the  $k^2$ -dependence in Equation (20), and indeed for all  $\nu > \frac{1}{2}$ , means that for these values of  $\nu$  the odd arrangement of  $\delta'$  interactions does not converge to a generalized point interaction.

### 3. TUNNELING TIMES FOR BARRIERS WITH WELL-DEFINED SYMMETRY

In this section we will consider the tunneling times for a non-relativistic particle scattered by two point barriers. We are mainly interested in the time associated with the transmission across the double barrier as given by both a SWP clock, which runs only when the particle is in this region, and by the phase time.

In order to calculate the SWP clock time, it is sufficient for the purposes of this work to notice that the effect of the particle-clock coupling is equivalent to introduce a weak potential  $V(x)$  equal to a constant  $V_0$  ( $V_0 \ll E$ ) in the region  $(x_1, x_2)$  and zero otherwise (for an in depth account of the SWP clock see [12, 13]). Introducing this coupling into the Schrödinger Equation (1) it becomes

$$\psi''(x) + k^2 \psi = s_1 [\psi](x) + s_2 [\psi](x) + \theta(x - x_1) \theta(x_2 - x) V_0 \psi(x), \quad (21)$$

where  $\theta(x)$  is the Heaviside theta function.

Assuming a particle incident upon the barriers from the left, the reflection and transmission amplitudes  $[R(k, q)$  and  $T(k, q)$ , respectively] can be immediately obtained from the formalism developed in Section 2. Then, by considering a unit incident flux, we have  $A_1 = 1, A_2 = R(k, q), C_1 = T(k, q)$  and  $C_2 = 0$ , and using Equation (6) the incident and reflection amplitudes can be written in terms of the matrix elements of Equation (7) as

<sup>5</sup>For details the reader is referred to Calçada et al. (in preparation).



$$R(k, q) = \frac{-\mathcal{M}_{21}}{\mathcal{M}_{22}} \quad \text{and} \quad T(k, q) = \frac{e^{2i(\theta_1 + \theta_2)}}{\mathcal{M}_{22}}, \quad (22)$$

where in the last equation we have used the fact that  $\det \mathcal{M} = e^{2i(\theta_1 + \theta_2)}$ , as can be promptly verified from the conditions on the parameters in Equation (4) and from Equations (7) to (9).

For the purpose of calculating tunneling times the transmission and reflection amplitudes given by Equation (22), with  $k \equiv \sqrt{E}$  and  $q \equiv \sqrt{k^2 - V_0}$ , are more conveniently expressed as

$$\begin{aligned} T(k, q) &= |T(k, q)| e^{i[\varphi_T(k, q) - k(x_2 - x_1)]} \quad \text{and} \\ R(k, q) &= |R(k, q)| e^{i[\varphi_R(k, q) + 2kx_1]}, \end{aligned} \quad (23)$$

where  $\varphi_T(k, q)$  is the phase difference, at any time  $t$ , between the transmitted wave at  $x_2$  and the incident wave function at  $x_1$ , whereas  $\varphi_R(k, q)$  is the phase difference, at any time  $t$ , between the reflected and the incident waves at  $x_1$ . Then, the SWP clock times for transmission and reflection (sometimes called the *conditional dwell times* [21]) are given by [12, 13]

$$t_c^{T(R)}(k) = -\hbar \left[ \frac{\partial}{\partial V_0} \varphi_{T(R)}(k, q) \right]_{V_0=0} \quad (24)$$

and satisfy the identity  $\tau_D(k) = |T(k, k)|^2 t_c^T(k) + |R(k, k)|^2 t_c^R(k)$ , where  $\tau_D(k)$  is the dwell time [13].

Similarly, the transmission and reflection *phase times* are given by [10, 15]

$$t_p^{T(R)}(k) = \hbar \frac{\partial}{\partial E} \varphi_{T(R)}(k, k). \quad (25)$$

Since the point interaction parameters  $\theta_1$  and  $\theta_2$  cause a simple shift in the phase difference, and due to its independence of the energy  $E$  and the perturbation potential  $V_0$ , it follows that both the clock and the phase times are independent of these parameters—for even and odd arrangements, Equations (13) and (15), there is, in fact, no change in phase due to  $\theta_1$  and  $\theta_2$  (an in-depth analysis of the consequences of the phase factor, arising at the boundary conditions, for one-dimensional quantum tunneling is presented in Furuhashi et al. [59]). In what follows we consider only arrangements of point interactions with well-defined symmetry under parity transformations, according to Equations (13)–(16), and obtain explicit results for the clock and phase times. The tunneling times are further analyzed for special cases of one-parameter arrangements.

### 3.1. Even Arrangements

From Equation (22) we have that for even arrangements of two point interactions the difference between the phases  $\varphi_R(k, q)$  and  $\varphi_T(k, q)$  is given by

$$\varphi_R(k, q) - \varphi_T(k, q) = \arg(-\mathcal{M}_{21}) = \pm \frac{\pi}{2}, \quad (26)$$

since in this case  $\mathcal{M}_{21}$  is purely imaginary

$$\mathcal{M}_{21} = i \frac{\sin(2Lq) [-q^2 (a^2 + b^2 k^2) + c^2 + d^2 k^2] + 2q \cos(2Lq) (ac + bdk^2)}{2kq}. \quad (27)$$

The result expressed in Equation (26) generalizes the well-known result for symmetric (even) regular potentials [10] to the case of an even arrangement of two point interactions. It follows that in this case  $t_p^{(T)}(k) = t_p^{(R)}(k)$ , and from Equations (26) together with the relationship between dwell and SWP times [13] we have  $t_c^{(T)}(k) = t_c^{(R)}(k) = \tau_D(k)$ .

The general expressions for the clock and phase times for an arbitrary *even* double point interaction are

$$t_c^{(T)}(k) = t_c^{(R)}(k) = \frac{N_c(k)}{D(k)}, \quad (28)$$

$$t_p^{(T)}(k) = t_p^{(R)}(k) = \frac{1}{k} \frac{N_p(k)}{D(k)}, \quad (29)$$

where we defined the following functions

$$\begin{aligned} N_c(k) &= 4k(ac + bdk^2) \sin^2(2kL) - [c^2 - k^2(a^2 + b^2 k^2 - d^2)] \\ &\quad \times \sin(4kL) + 4kL[k^2(a^2 + d^2) + b^2 k^4 + c^2], \end{aligned}$$

$$\begin{aligned} D(k) &= 2c^2 k^2 (a^2 + d^2) + k^4 [a^4 + 2a^2(b^2 k^2 + 2d^2) - 4ad \\ &\quad + (b^2 k^2 + d^2)^2 + 6] - \cos(4kL) \{2c^2 k^2 (d^2 - 3a^2) \\ &\quad + k^4 [a^4 + 2b^2 k^2 (a^2 - 3d^2) - 12a^2 d^2 + 12ad + b^4 k^4 \\ &\quad + d^4 - 2] + c^4\} + 4k \sin(4kL) [-ck^2 (a^3 - 2ad^2 + d) \\ &\quad + bk^4 (-2a^2 d + a + d^3) + ac^3 - b^3 dk^6] + c^4, \end{aligned}$$

$$\begin{aligned} N_p(k) &= -\cos(4kL) [bk^4 (a^3 + ab^2 k^2 - 6ad^2 + 5d) + ck^2 \\ &\quad \times (-6a^2 d + 5a + d^3) + c^3 d] - 2k \sin(4kL) [(2ad - 1) \\ &\quad \times (b^2 k^4 - c^2) + 2k^2 (a^2 - d^2) (ad - 1)] + k^4 \\ &\quad \times [b(a^3 + ab^2 k^2 + 2ad^2 + d) + 4L(a^2 + b^2 k^2 + d^2)] \\ &\quad + c^3 d + 4c^2 k^2 L + ck^2 (2a^2 d + a + d^3). \end{aligned}$$

Even though Equations (28) and (29) represent the most general expressions for the corresponding tunneling times with a potential barrier given by an even arrangement of two point interactions, for further analysis of the behavior of these times it is necessary to consider specific interactions. Thus, let us now consider three special and important cases of one-parameter even arrangements—in all the cases below the parameter  $\gamma \in \mathbb{R}$  is the “strength” of the interaction.

(i) The first case to be considered is that of an even arrangement of two delta ( $\delta$ ) interactions, which corresponds to set  $\theta = 0$ ,  $a = d = 1$ ,  $b = 0$  and  $c = \gamma$  in Equations (28)–(29). In this case, the clock and phase times for reflection and transmission are given by

$$\begin{aligned} t_c^{(T)}(k) &= t_c^{(R)}(k) \\ &= \frac{2k(\gamma + 4k^2 L + 2\gamma^2 L) - \gamma^2 \sin(4kL) - 2\gamma k \cos(4kL)}{\gamma^4 + 8k^4 + 4\gamma^2 k^2 - \gamma^2 (\gamma^2 - 4k^2) \cos(4kL) + 4\gamma^3 k \sin(4kL)}, \end{aligned} \quad (30)$$

$$\begin{aligned}
t_p^{(T)}(k) &= t_p^{(R)}(k) \\
&= \frac{\gamma^3 + 8k^4L + 4\gamma k^2 + 4\gamma^2 k^2 L - \gamma^3 \cos(4kL) + 2\gamma^2 k \sin(4kL)}{k[\gamma^4 + 8k^4 + 4\gamma^2 k^2 - \gamma^2(\gamma^2 - 4k^2) \cos(4kL) + 4\gamma^3 k \sin(4kL)]}.
\end{aligned} \quad (31)$$

The expression above for the clock (dwell) time agrees with previous results [21, 23, 24].

As previously mentioned, in this work we are particularly interested in the behavior of the tunneling times in the opaque regime (i.e., for  $|T|^2 \ll 1$  and  $|R|^2 \simeq 1$ ). For the even double- $\delta$  interaction this regime corresponds to take  $|\gamma| \gg 1$ . It is straightforward to verify from Equations (30) to (31) that  $t_c^{(T)}(k) \rightarrow 0$  and  $t_p^{(T)}(k) \rightarrow 0$  in the extreme opaque limit  $|\gamma| \rightarrow \infty$ , which is usually taken as an indication of the existence of the GHE [21, 23, 24]. However, **Figure 1A** shows that the off-resonance values of both the phase and clock (dwell) times clearly increase with the barrier separation—for the parameters in **Figure 1A** the scattering is well inside the opaque regime—thus, contradicting the GHE, according to which one would expect the off-resonance times to be independent of the barrier spacing.

In order to better understand the behavior of the tunneling times when the interaction approaches the opaque limit, it is important to consider their asymptotic expressions when  $|\gamma| \gg 1$ . Up to the second order in  $\gamma^{-1}$  we obtain

$$t_c^{(T)}(k) \sim \left(\frac{1}{\gamma}\right)^2 [2kL \csc^2(2kL) - \cot(2kL)], \quad (32)$$

$$t_p^{(T)}(k) \sim \frac{1}{\gamma k} + \left(\frac{1}{\gamma}\right)^2 [2kL \csc^2(2kL) - 2 \cot(2kL)]. \quad (33)$$

These expressions indicate that for any *finite*  $\gamma$ , such that  $|\gamma| \gg 1$ , both the clock and the phase times depend on the barrier spacing  $L$  at the same order  $\left(\frac{1}{\gamma}\right)^2$ . For a fixed value of the strength  $\gamma$ , and apart from the oscillating terms, such a dependence tends to grow linearly with  $L$  in the same fashion for both times. Thus, it is strictly only for  $\gamma = \infty$  (at which point the tunneling times lose their physical meaning because the barriers become impenetrable) that the above expressions become independent of  $L$  (since they are zero).

(ii) The second case to be considered is an even arrangement of two “delta prime” ( $\delta'$ ) interactions, which corresponds to the choices  $\theta = 0, a = d = 1, b = \gamma$  and  $c = 0$  in Equations (28)–(29). The resulting clock and phase times are

$$\begin{aligned}
t_c^{(T)}(k) &= t_c^{(R)}(k) \\
&= \frac{2(\gamma + 2\gamma^2 k^2 L + 4L) + \gamma^2 k \sin(4kL) - 2\gamma \cos(4kL)}{k[\gamma^4 k^4 - 4\gamma^3 k^3 \sin(4kL) + 4\gamma^2 k^2 - \gamma^2 k^2 (\gamma^2 k^2 - 4) \cos(4kL) + 8]},
\end{aligned} \quad (34)$$

$$\begin{aligned}
t_p^{(T)}(k) &= t_p^{(R)}(k) \\
&= \frac{4\gamma + \gamma^3 k^2 - \gamma^3 k^2 \cos(4kL) + 4\gamma^2 k^2 L - 2\gamma^2 k \sin(4kL) + 8L}{k[\gamma^4 k^4 - 4\gamma^3 k^3 \sin(4kL) + 4\gamma^2 k^2 - \gamma^2 k^2 (\gamma^2 k^2 - 4) \cos(4kL) + 8]}.
\end{aligned} \quad (35)$$

The opaque regime is again given by  $|\gamma| \gg 1$ , and in the limit  $|\gamma| \rightarrow \infty$  both Equations (34) and (35) tend to zero. However, similarly to the previous case, a plot of the phase and clock times, for parameters well inside the opaque regime, shows that off-resonance both times depend quasi-linearly on the barrier spacing—see **Figure 1B**.

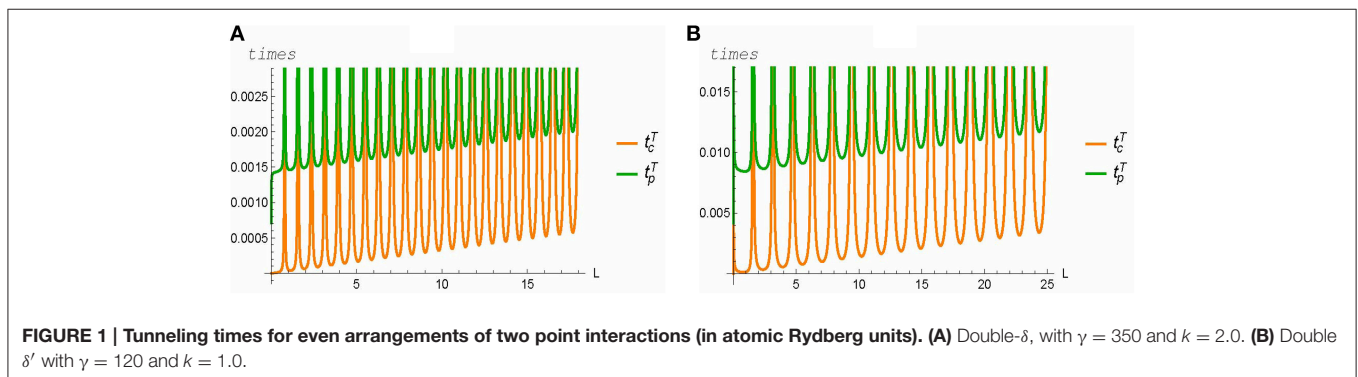
The asymptotic expressions for the times in Equations (34) and (35), in the opaque regime  $|\gamma| \gg 1$  are given, respectively, by

$$t_c^{(T)}(k) \sim \left(\frac{1}{\gamma}\right)^2 \frac{1}{k^4} [2kL \csc^2(2kL) + \cot(2kL)], \quad (36)$$

$$t_p^{(T)}(k) \sim \frac{1}{\gamma k^3} + \left(\frac{1}{\gamma}\right)^2 \frac{1}{k^4} [2kL \csc^2(2kL) + 2 \cot(2kL)], \quad (37)$$

up to second order in  $1/\gamma$ . From these expressions it is clear that for any large, but *finite*, interaction strength  $\gamma$  the off-resonance tunneling times tend to increase linearly with the barrier spacing.

(iii) The third special case that we consider here is an even arrangement of two so-called  $\delta^{(1)}$  interactions [53], which corresponds to choose  $\theta = 0, a = \gamma, b = 0, c = 0$  and  $d = \gamma^{-1}$  in Equations (28)–(29). Notice that a single  $\delta^{(1)}$  interaction does *not* have a defined symmetry under parity transformations (contrary to the single  $\delta$  and  $\delta'$  interactions, which are even under



parity)—see e.g., [39] for a parity-symmetry analysis for single interactions. The clock and phase times, for the even arrangement of  $\delta^{(1)}$ s, are

$$t_c^{(T)}(k) = t_c^{(R)}(k) = \frac{\gamma^2}{k^2} \frac{4(\gamma^4 + 1)kL + (\gamma^4 - 1)\sin(4kL)}{\gamma^8 + 6\gamma^4 - (\gamma^4 - 1)^2 \cos(4kL) + 1}, \quad (38)$$

$$t_p^{(T)}(k) = t_p^{(R)}(k) = \frac{4\gamma^2 L}{k} \frac{(\gamma^4 + 1)}{\gamma^8 + 6\gamma^4 - (\gamma^4 - 1)^2 \cos(4kL) + 1}. \quad (39)$$

It should be noticed that the phase time is invariant under the substitution  $\gamma \rightarrow \gamma^{-1}$ , while the clock time does not present such symmetry. This follows from the absence of a defined parity for  $\delta^{(1)}$ , as noted above, and the fact that the change  $\gamma \rightarrow \gamma^{-1}$  in effect merely swaps the location of the two (different) point interactions. The phase time, being the “temporal delay” caused by the interactions [60], is insensitive to this location swapping—as long as the interbarrier distance remains the same. On the other hand, since the clock (in reality a stop-watch) runs only while there is a probability of finding the particle in between the barriers, and since  $\delta^{(1)}$  is not even, such probability depends on the order of the two interactions, and the clock (dwell) time is not expected to be invariant under the substitution  $\gamma \rightarrow \gamma^{-1}$ .

The opaque regime for the even arrangement of  $\delta^{(1)}$  interactions, similarly to the single  $\delta^{(1)}$  case, is given by two different ranges of the interaction strength, namely,  $|\gamma| \ll 1$  and  $|\gamma| \gg 1$  [61]<sup>6</sup>. In **Figure 2** we present a typical plot of both times in the regime  $|\gamma| \ll 1$  (plots for the regime  $|\gamma| \gg 1$  have similar characteristics). As it can be seen, in the opaque regime the phase and clock times are essentially identical for  $L \gtrsim 1.2$ , with the differences being of the order  $\gamma^2$  (or  $1/\gamma^2$  in the regime  $|\gamma| \gg 1$ )—these minute differences between the two times in this regime, which come from the term  $(\gamma^2/k^2)(\gamma^4 - 1)\sin(4kL)/[\gamma^8 + 6\gamma^4 - (\gamma^4 - 1)^2 \cos(4kL) + 1]$  in Equation (38), can be seen in **Figure 2B**.

<sup>6</sup>In Gadella et al. [61] the  $\delta^{(1)}$  interaction is given by choosing  $a = 0$  and identifying  $\gamma = \frac{1+mb}{1-mb}$  in their Equation (22). Thus, from Equation (23) of the same reference, it follows that the transmission coefficient goes to zero (the opaque limit) when  $b \rightarrow -1/m$  and  $b \rightarrow +1/m$ , which in our notation correspond to  $\gamma \rightarrow 0$  and  $|\gamma| \rightarrow \infty$ , respectively.

As follows from **Figure 2A**, even in the opaque regime both times do depend on  $L$ , as was the case with the other arrangements considered.

The dependence of the tunneling times on  $L$  is, once again, clear when one considers the asymptotic behavior of the times in the opaque region. For  $|\gamma| \gg 1$ , and up to second order in  $\frac{1}{\gamma}$ , the asymptotic expansion for the tunneling times are given by

$$t_c^{(T)}(k) \sim \left(\frac{1}{\gamma}\right)^2 \frac{1}{k^2} [2kL \csc^2(2kL) + \cot(2kL)], \quad (40)$$

$$t_p^{(T)}(k) \sim \left(\frac{1}{\gamma}\right)^2 \frac{2L \csc^2(2kL)}{k}, \quad (41)$$

and, from the above mentioned invariance of the phase time, the asymptotic behavior of Equation (39) when  $\gamma \ll 1$  can be obtained from Equation (41) by simply changing  $\gamma \rightarrow \gamma^{-1}$ , whereas the corresponding asymptotic behavior of the clock time can be obtained from Equation (40) by changing  $\gamma \rightarrow \gamma^{-1}$  and by replacing a minus sign in front of the “cot” term. It follows from Equations (40) to (41) that for any *finite* (conversely, *different of zero*)  $\gamma$  the tunneling times will depend on  $L$  (again, this dependence tends to be linear for off-resonance times).

### 3.2. Odd Arrangements

Let us now consider odd arrangements of two point interactions. From Equations (15), (22), and (23) the difference between the reflection and transmission phases is

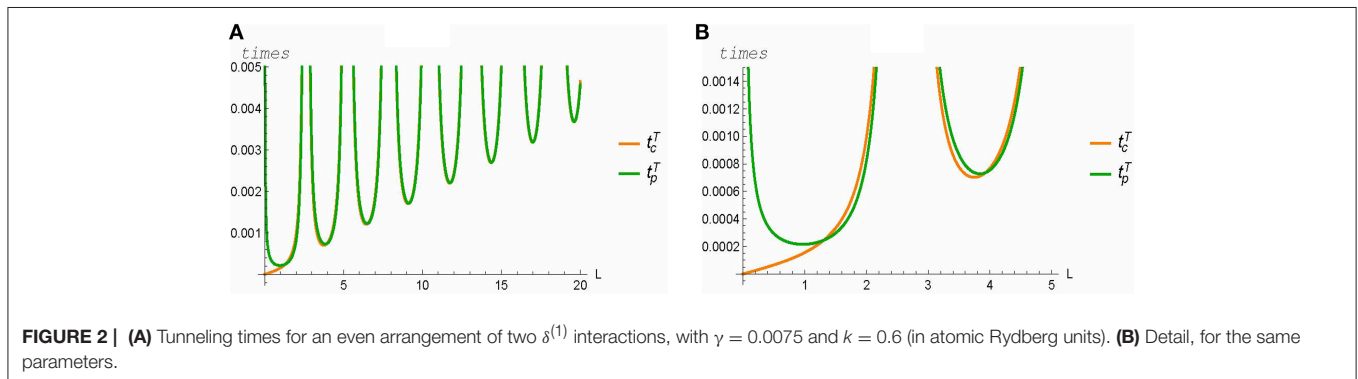
$$\varphi_R(k, q) - \varphi_T(k, q) = \arg(-\mathcal{M}_{21}) = \tan^{-1} \left[ \frac{k^2(b^2 q^2 + 1) - c^2 - q^2}{2k(bq^2 + c)} \right] \equiv \varphi_0(k, q), \quad (42)$$

where we have defined the auxiliary phase  $\varphi_0(k, q)$ , which allow us to introduce the auxiliary clock and phase times [62, 63] given, respectively, by

$$t_c^0(k) \equiv -\hbar \left[ \frac{\partial}{\partial V_0} \varphi_0(k, q) \right]_{V_0=0} \quad \text{and} \quad t_p^0(k) \equiv \hbar \frac{\partial}{\partial E} \varphi_0(k, k).$$

With the help of these auxiliary times, the transmission and reflection times are now related by

$$t_c^{(R)}(k) = t_c^{(T)}(k) + t_c^0(k) \quad \text{and} \quad t_p^{(R)}(k) = t_p^{(T)}(k) + t_p^0(k). \quad (43)$$



**FIGURE 2 | (A)** Tunneling times for an even arrangement of two  $\delta^{(1)}$  interactions, with  $\gamma = 0.0075$  and  $k = 0.6$  (in atomic Rydberg units). **(B)** Detail, for the same parameters.

It should be noticed that, contrary to the case of even potentials, for odd potentials (and potentials without a defined parity) the clock times in general do *not* coincide with the dwell time [63].

Below we consider two cases of odd arrangements of point interactions, the first built from two  $\delta$  and the second built from two  $\delta'$  interactions. We note that it is not possible to build odd arrangements with two  $\delta^{(1)}$  point interactions.

(i) We first consider the odd arrangement of two  $\delta$  interactions, which corresponds to set  $b = 0$  in Equation (16). Then, the strength of the interaction is  $c \equiv \gamma$  and the transmission and auxiliary times are given, respectively, by

$$t_c^{(T)}(k) = \frac{2N_1(k) - N_2(k)}{2D(k)}, \quad (44)$$

$$t_p^{(T)}(k) = \frac{N_1(k) - N_2(k)}{D(k)}; \quad (45)$$

$$t_c^0(k) = \frac{-2k}{\gamma} \frac{1}{D_0(k)}; \quad (46)$$

$$t_p^0(k) = \frac{\gamma}{k} \frac{1}{D_0(k)}, \quad (47)$$

where we have defined the functions

$$N_1(k) = 2kL(\gamma^2 + 2k^2);$$

$$N_2(k) = \gamma^2 \sin(4kL);$$

$$D(k) = \frac{1}{2} [\gamma^4 + 8k^4 + 4\gamma^2 k^2 - \gamma^2 (\gamma^2 + 4k^2) \cos(4kL)];$$

$$D_0(k) = 4k^2 + \gamma^2.$$

The corresponding reflection times follow directly from these expressions together with Equation (43).

The opaque regime is characterized by  $|\gamma| \gg 1$ , and it is straightforward to see that all times in Equations (44)–(47) tend to zero in the extreme opaque limit  $|\gamma| \rightarrow \infty$ . **Figure 3A** shows a plot of the transmission times for an interaction strength well inside the opaque regime ( $\gamma = 80$ ). The difference between the clock and phase times,  $t_c - t_p = N_2(k)/[2D(k)]$ , behaves as  $1/\gamma^2$  in this regime and it is not noticeable in the figure, except for very small  $L$ . **Figure 3A** shows that both the off-resonance phase and clock times increase with the interbarrier spacing in this opaque regime; in addition, similarly to the cases previously

considered, the behavior of these times is essentially linear. Again, this dependence on  $L$  for any finite interaction strength is made clear by the asymptotic expansion of the times for  $|\gamma| \gg 1$ . Up to second order in  $\frac{1}{\gamma}$ , these asymptotic expansions are:

$$t_c^{(T)}(k) \sim \left(\frac{1}{\gamma}\right)^2 [2kL \csc^2(2kL) - \cot(2kL)]; \quad (48)$$

$$t_p^{(T)}(k) \sim \left(\frac{1}{\gamma}\right)^2 [2kL \csc^2(2kL) - 2 \cot(2kL)]; \quad (49)$$

$$t_c^0(k) \sim 0; \quad (50)$$

$$t_p^0(k) \sim \frac{1}{\gamma k}, \quad (51)$$

and it is clear that, strictly, only for  $\gamma = \infty$  there is no  $L$  dependence.

(ii) The second possibility we will consider for an odd arrangement of two point interactions is to set  $b = \gamma$  and  $c = 0$  in Equation (16), which corresponds to an odd arrangement of two  $\delta'$  interactions. In this case, the transmission and auxiliary clock and phase times are given by

$$t_c^{(T)}(k) = \frac{2M_1(k) + M_2(k)}{2k\Delta(k)}, \quad (52)$$

$$t_p^{(T)}(k) = \frac{M_1(k) + M_2(k)}{k\Delta(k)}; \quad (53)$$

$$t_c^0(k) = \frac{-2}{k^3\gamma} \frac{1}{\Delta_0(k)}; \quad (54)$$

$$t_p^0(k) = \frac{\gamma}{k} \frac{1}{\Delta_0(k)}, \quad (55)$$

where

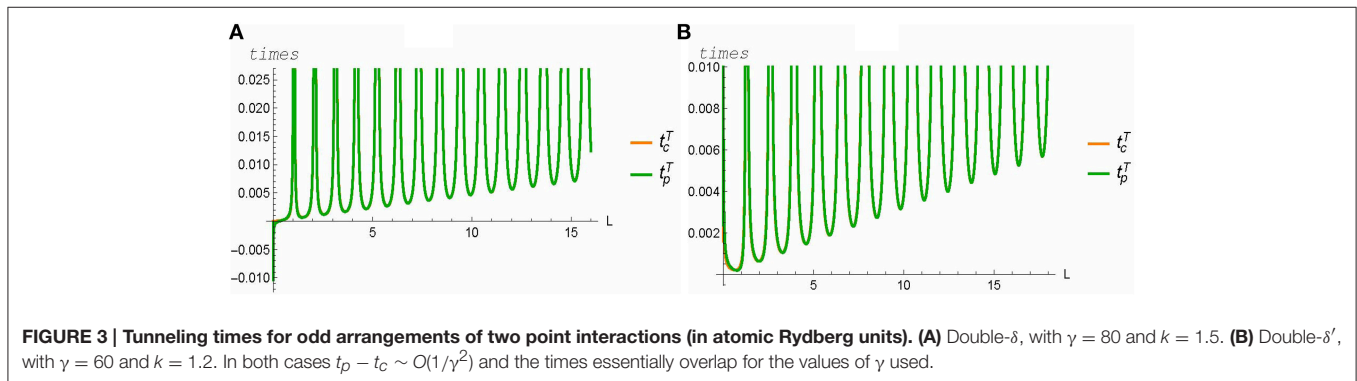
$$M_1(k) = 2L(\gamma^2 k^2 + 2);$$

$$M_2(k) = \gamma^2 k \sin(4kL);$$

$$\Delta(k) = \frac{1}{2} [\gamma^4 k^4 + 4\gamma^2 k^2 - \gamma^2 k^2 (\gamma^2 k^2 + 4) \cos(4kL) + 8];$$

$$\Delta_0(k) = 4 + k^2 \gamma^2.$$

As before, the opaque regime is characterized by  $\gamma \gg 1$  and in the extreme opaque limit,  $|\gamma| \rightarrow \infty$ , all the times above go to zero.





In what regards the dependence of the transmission tunneling times with  $L$ , in the opaque regime, and its consequences to the GHE, the behavior of the times is similar to all the previous cases considered, as can be seen from **Figure 3B**, which clearly shows that the off-resonance times tend to increase linearly with  $L$  (for this interaction  $t_p - t_c = M2(k)/[2\Delta(k)] \sim O(\frac{1}{\gamma^2})$  in the opaque regime. Hence the apparent coincidence of the times in **Figure 3B**). Once again, this behavior can be understood from the asymptotic expansions of the times for  $|\gamma| \gg 1$ , which are

$$t_c^{(T)}(k) \sim \left(\frac{1}{\gamma}\right)^2 \frac{1}{k^4} [2kL \csc^2(2kL) + \cot(2kL)]; \quad (56)$$

$$t_p^{(T)}(k) \sim \left(\frac{1}{\gamma}\right)^2 \frac{1}{k^4} [2kL \csc^2(2kL) + 2 \cot(2kL)]; \quad (57)$$

$$t_c^0(k) \sim 0; \quad (58)$$

$$t_p^0(k) \sim \frac{1}{\gamma k^3}. \quad (59)$$

and which depend on  $L$  for any *finite*  $\gamma$  (the expansions for the auxiliary times are not necessary in the analysis of the GHE and are presented only for completeness).

## 4. CONCLUDING REMARKS

An analysis of the symmetry under parity transformation and of the tunneling times was provided for a one-dimensional non-relativistic particle scattered by a double barrier composed of two general point potentials.

Our general analysis of the symmetry shows that it is possible to construct potentials with well-defined parity, both odd and even, for such double barriers of point interactions. This is in stark contrast with the case of a single point interaction, for which no odd interaction exists [37, 39, 64]—as illustrated in Section 2 by the fact that the limit  $L \rightarrow 0$  of a renormalized  $\delta$ -dipole interaction, an odd arrangement of two  $\delta$  [35], is either trivial or results in a point interaction which is even or has no defined parity. Furthermore, we extended Šeba's [35] procedure to the case of a renormalized odd arrangement of  $\delta'$  interactions and found that in the limit  $L \rightarrow 0$  this arrangement is either trivial (for  $\nu < 1/2$ ) or it does not converge to a general singular interaction (for  $\nu \geq 1/2$ ). In other words, for  $\nu \geq 1/2$  such limit is not of the form (Equation 2) [equivalently, (Equation 4)] and has no self-adjoint extension as well.

We then proceeded to calculate the quantum tunneling times for the double point barriers with well defined parity. We calculated the phase and SWP clock times for three special cases of even arrangements [pairs of  $\delta$ ,  $\delta'$  or  $\delta^{(1)}$  satisfying the constraints (Equations 13 and 14)] as well as for two odd arrangements of double point barriers [pairs of  $\delta$  or

$\delta'$  satisfying (Equation 15)–(Equation 16)]. Even though the reflection tunneling times were also provided for completeness, our main concern was the behavior of the transmission times in the opaque regime and the existence of the so-called generalized Hartman effect [16].

We found that, for all interactions considered, both the phase and SWP clock times approach zero in the extreme opaque limit. However, in all cases, for any *finite* values of the interaction strength (for the even arrangement of  $\delta^{(1)}$ s, also  $\gamma \neq 0$ ) the tunneling times do depend on the interbarrier spacing, and such dependence is essentially linear for off-resonance values of the parameters. It is only strictly for the interaction strength at the extreme opaque value ( $|\gamma| = \infty$  or  $\gamma = 0$  for the even double- $\delta^{(1)}$ ;  $|\gamma| = \infty$  for all other interactions) that the times do not depend on  $L$ , since in this case their off-resonance value is zero. However, for these extreme values of the parameters the very concept of tunneling time loses its meaning, since there is no transmission (the barriers become impenetrable and thus the solutions in both sides of each barrier are separable). Therefore, for the potentials analyzed the GHE is just a mathematical artifact of such an extreme limit and devoid of any physical meaning. As a final remark we stress that even if the GHE is an artifact, its more striking consequence, which is the apparent superluminality of tunneling, is not precluded—in fact, superluminality does not require GHE (or HE), and the slow growth of the tunneling times with respect to the interbarrier distance in the opaque regime could result, for sufficiently separated barriers, in an apparent superluminal transmission. However, any apparent violation of causality is avoided by interpreting the superluminal phase and clock times respectively in terms of wave packet reshaping and weak measurements [21, 23] (for a recent treatment of tunneling times from the point of view of weak measurement theory see [65]). It should also be noticed that apparent superluminality is more properly addressed in a relativistic theory. The above results on the GHE reinforce similar conclusions reached in Winful [1], Lunardi and Manzoni [20], Aharonov et al. [21] and Lunardi et al. [24]—an analysis of the GHE is, unavoidably, case-by-case and, thus, this work adds to the body of results clarifying such effect.

## AUTHOR CONTRIBUTIONS

JL and LM contributed equally and in all stages of this work. ML and EN contributed in the calculation of tunneling times.

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**Conflict of Interest Statement:** The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

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